1. Multiple choice questions.

a). (5 *points*) Which of the following sets are vector spaces? (   )

(A) $\{(a, b) \in \mathbb{R}^2 : \ b = 2a + 3\} \subseteq \mathbb{R}^2$, with the usual " + " and " · " as in $\mathbb{R}^2$.

(B) $\{\mathbf{v} \in \mathbb{R}^3 : \ ||\mathbf{v}|| = 1\} \subseteq \mathbb{R}^3$, with the usual " + " and " · " as in $\mathbb{R}^3$.

(C) {All polynomials in P_2 that are divisible by $x - 2$}, with the usual " + " and " · " as in P_2.

(D) The set $\mathbb{R}^2$, with addition and scalar multiplication given by: for $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $k \in \mathbb{R}$, $\mathbf{x} + \mathbf{y} := (x_1 + 2y_1, \ x_2 + 3y_2)$, $k\mathbf{x} := (kx_1, kx_2)$.

b). (5 *points*) Determine which of the following statements are true. (   )

(A) If $A \in \mathbb{M}_{n \times n}$ is invertible, then its adjoint $adj(A)$ is also invertible.

(B) Let $E \in \mathbb{M}_{3\times3}$ be an elementary matrix such that $det(E) = 1$, then E must be the identity matrix in $\mathbb{M}_{3\times3}$.

(C) Let $V \subsetneq \mathbb{R}^5$ be a subspace, then any set of five vectors in V is linearly dependent.

(D) If $A \in \mathbb{M}_{4\times7}$, and $dim\,(null(A)) = 3$, then for all $\mathbf{b} \in \mathbb{R}^4$, the linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution.

c). (5 *points*) Consider a linearly independent set $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subseteq V$ for some $m \geq 1$, and let $\mathbf{v} \in V$. Which possible values can $dim\,(span\{\mathbf{v}_1 + \mathbf{v}, \cdots, \mathbf{v}_m + \mathbf{v}\})$ take? (   )

(*A*) $m-1$ (*B*) m (*C*) $m+1$ (*D*) $m+2$

2. Fill in the blanks.

a.) (5 *points*) Let $A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$. Then $(adj(A))^{-1} =$ ________.

b). (5 *points*) Let $B = \{1, x, x^2\}$ and $B' = \{1 + x^2, x + x^2, 1 + 2x + x^2\}$ be two basis for P_2.

Then the transition matrix $P_{B' \leftarrow B}$ from B to B' is ________.

c.) (5 *points*) Let $A = [a_{ij}] \in \mathbb{M}_{n\times n}$ be given such that $a_{ij} = ij$ for all $i, j = 1, \cdots, n$. Assuming that $n \geq 2$, then $det\,A =$ ________.

3. (10 *points*) Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, and suppose that $A^2 - AB = I_3$. Find B.

4. Let $A \in \mathbb{M}_{4\times 5}$ be the following matrix

$$\begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & 3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}$$

a). (10 *points*) Compute $r(A), nullity(A)$, and find basis for $row\,(A), col\,(A)$ and $null(A)$.

b). (5 *points*) Determine whether $\mathbf{u} = [2, 1, 7, -12]^T$ belongs to $col(A)$.

c). (5 *points*) Find the space of all vectors in $\mathbb{R}^4$ that are orthogonal to $col(A)$, i.e. the *orthogonal complement* of $col(A)$ in $\mathbb{R}^4$.

5. Let $\mathbb{M}_{2\times 2}$ denote the vector space of all 2×2 matrices with real entries. Consider the following two subsets of $\mathbb{M}_{2\times 2}$

$$U = \left\{ \begin{bmatrix} x & -x \\ y & z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}; \qquad W = \left\{ \begin{bmatrix} a & b \\ -a & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

a). (10 *points*) Verify that both U and W are vector subspaces of $\mathbb{M}_{2\times 2}$. And find a basis and the dimension of U and W.

b). (10 *points*) Find the dimensions and basis of the subspaces $U + W$ and $U \cap W$.

7. a). (5 *points*) Let $\mathbf{v}_1 = [1,3,0,2]^T$, $\mathbf{v}_2 = [-1,0,1,0]^T$, $\mathbf{v}_3 = [5,9,-2,6]^T$ be vectors in $\mathbb{R}^4$. Is it possible to find a set of numbers $\{a_{ij} \mid i,j = 1,2,3\}$, such that the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent? Here $\mathbf{w}_i$'s are given by

$$\begin{aligned}
\mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + a_{13}\mathbf{v}_3 \\
\mathbf{w}_2 &= a_{21}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + a_{23}\mathbf{v}_3 \\
\mathbf{w}_3 &= a_{31}\mathbf{v}_1 + a_{32}\mathbf{v}_2 + a_{33}\mathbf{v}_3
\end{aligned}$$

Please give full explanation of your claim.

b). (5 *points*) You should have already known the fact (from the review problems) that a matrix of of the form $A = \mathbf{u}\mathbf{v}^T$ has rank 1, here $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are any n dimensional non zero column vectors. What about the converse? That is, is it true that any rank 1 square matrix of size n can be written as $\mathbf{u}\mathbf{v}^T$ for some n dimensional non zero column vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$? Prove your claim.

8. (10 *points*) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ be two linearly independent sets of vectors in $\mathbb{R}^n$ for some integer n such that $\mathbf{v}_i \bullet \mathbf{w}_j = 0$ for all $i = 1, 2, 3$ and $j = 1, 2$. Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2\}$ still linearly independent? Verify your claim.