Lecture Notes (Linear Algebra I: MATH1112.06)

December 6, 2023

1 Midterm Exam 8:15-9:55

1.1 1.(a) Answer: C

Which of the following sets are vector spaces?

1. $\{(a,b) \in \mathbb{R} \colon b = 2a+3\} \subset \mathbb{R}^2$, with usual $+, \cdot$ as in $\mathbb{R}^2$ False, not closed under scaling by 0.
2. $\{\mathbf{v} \in \mathbb{R}^3 \colon |v| = 1\}$ with usual $+$ and $\times$ in $\mathbb{R}^3$. False, not closed under scaling by 0
3. All polynomials in P_2 that are divisible by $x-2$, with usual $+$ and $\cdot$ in P_2.True.
4. The set $\mathbb{R}^2$ with scaling given by
$$k(x_1, x_2) = (kx_1, kx_2)$$
and addition
$$(x_1, x_2) + (y_1, y_2) = (x_1 + 2y_1, x_2 + 3y_2).$$
False
$$(0,0) = 0(x_1,x_2) = (1+(-1))(x_1,x_2) = (x_1,x_2)+(-x_1-x_2) = (x_2-2x_1, x_2-3x_2) = (-x_1, -2x_2) \neq (0,0)$$

1.2 1.(b) Answer: A,C,D

Determine which of the following statements are true?

1. If $A \in \mathbb{M}_{n\times n}$ is invertible, then its adjoint $\text{adj}(A)$ is also invertible. True.
2. Let $E \in \mathbb{M}_{3\times 3}$ be an elementary matrix such that $\det(E) = 1$, then E must be the identity matrix in $\mathbb{M}_{3\times 3}$. False, one can take $E = \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$.
3. Let $V \subsetneq \mathbb{R}^5$ be a subspace, then any set of five vectors in V is linearly dependent. . True.
4. If $A \in \mathbb{M}_{4\times 7}$ and $\dim(\text{Null}(A)) = 3$, then for all $\mathbf{b} \in \mathbb{R}^4$, the linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution. True. By rank-null equality, $\dim(\text{Col}(A)) = 7 - 3 = 4$, hence $\text{Col}(A) = \mathbb{R}^4$.

1.3 1.(c) Answer: A,B

Consider a linearly independent set $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subseteq V$ for some $m \geq 1$, and let $\mathbf{v} \in V$, which possible values can $\dim(\text{Span}\{\mathbf{v}_1 + \mathbf{v}, \cdots, \mathbf{v}_m + \mathbf{v}\})$ take?

1. $m-1$
2. m
3. $m+1$
4. $m+2$

Let us take $m = 1$, then $\dim(\mathrm{Span}\{\mathbf{v}_1 + \mathbf{v}\})$ can take 0, 1, two values, therefore the answers are $m, m-1$. For a strict proof: Let $V_1 = \mathrm{Span}\{\mathbf{v}_1 + \mathbf{v}, \cdots, \mathbf{v}_m + \mathbf{v}\}$, $V_2 = \mathrm{Span}\{\mathbf{v}\}$, then $V_1 + V_2 = \mathrm{Span}\{\mathbf{v}_1, \cdots \mathbf{v}_m, \mathbf{v}\}$, by the inclusion-exclusion principle, we have $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim V_1 \cap V_2$, so

$$\dim(\mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}) = \dim V_1 = \dim(V_1 + V_2) - 1 + \dim V_1 \cap V_2.$$

Now $\dim(V_1 + V_2) = m$ or $m+1$, according to whether $\mathbf{v} \in \mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ or not.

1. If $\mathbf{v} \in \mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$, then $\dim V_1 = m - 1 + \dim V_1 \cap V_2$, since V_2 is 1-dimensional, we have $\dim V_1 \cap V_2 \in \{0, 1\}$, the possible values for V_1 are $m-1$ and m. One could specifiy $\mathbf{v}$ to get these values, e.g., take $\mathbf{v} = -\mathbf{v}_1$ or $\mathbf{v} = \mathbf{0}$.
2. If $\mathbf{v} \notin \mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$, then $\dim V_1 = m + \dim V_1 \cap V_2$. Note that $\dim V_1 \cap V_2 \in \{0, 1\}$ but $\dim V_1 \leq m$ (since it is generated by m-elements), therefore $\dim V_1 = m$.

1.4 2.(a)

Let $A = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}$, THEN $(\mathrm{adj}(A))^{-1} =$? Note that $\mathrm{adj}(A) = \det(A) \cdot A^{-1}$, therefore $[\mathrm{adj}(A)]^{-1} = \det(A)^{-1} \cdot (A^{-1})^{-1} = \det(A)^{-1} \cdot A = \frac{1}{2} \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3/2 & 2 \end{pmatrix}$

1.5 2.(b)

Let $B = \{1, x, x^2\}$ and $B' = \{1 + x^2, x + x^2, 1 + 2x + x^2\}$ be two basis for P_2, then the transition matrix $P_{B' \leftarrow B}$ from B to B' is? By the notations and conventions on page 2, we it suffices to calculate the matrix P such that $(1, x, x^2) = (1 + x^2, x + x^2, 1 + 2x + x^2)P$. Let us rewrite the equation as

$$(1, x, x^2) = (1, x, x^2) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} P,$$

we know that $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ -1 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$

1.6 2.(c)

Let $A = [a_{ij}] \in \mathbb{M}_{n \times n}$ be given such that $a_{ij} = ij$ for all $i, j = 1, \cdots, n$, assuming $n \geq 2$, then $\det(A) = \mathbf{0}$. We calculate for $n = 2$, namely $\det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 0$ and saw in general the first two rows are proportional.

1.7

Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and suppose that $A^2 - AB = I_3$. Find B. Note that $A(A-B) = I_3$, therefore $A - B = A^{-1}$ so $B = A - A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

1.8 4.(a)

Let $\mathbf{M}_{4\times 5}$ be the following matrix $\begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & 3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{pmatrix}$. Compute $\mathrm{r}(A), \mathrm{nullity}(A)$ and basis for $\mathrm{Row}(A), \mathrm{Col}(A), \mathrm{Null}(A)$. We perform $S_{12}(-2), S_{13}(-3), S_{14}(-3)$ and get $\begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & -9 & 9 & -9 \\ 0 & 0 & -12 & 12 & -6 \end{pmatrix}$, then we perform $S_{24}(-6), S_2(-1/2), S_{23}(9), S_4(-2/9)$ to get $\begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ so

$$\mathrm{RREF}(A) = \begin{pmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this we see that $\mathrm{rank}(A) = 3, \mathrm{nullity}(A) = 2$, basis for $\mathrm{Row}(A)$ are given by the rows in RREF, which are

$$\{(1,3,0,3,0), (0,0,1,-1,0), (0,0,0,0,1)\},$$

a set of basis for $\mathrm{Col}(A)$ can be chosen to be pivot columns, namely

$$\{(1,2,3,3)^T, (4,6,3,0)^T, (2,3,-3,0)^t\}$$

, the null space $\mathrm{Null}(A) = \{(-3s-3t, s, t, t, 0)\} \colon s, t \in \mathbb{R}\}$ so a set of basis for $\mathrm{Null}(A)$ can be chosen as $\{(-3,1,0,0,0)^T, (-3,0,1,1,0)^T\}$

1.9

Determine whether $\mathbf{u} = (2,1,7,-12)^T$ belongs to $\mathrm{Col}(A)$.

We are asking whether the system of linear equation

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 6 \\ 3 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ = 1 \\ -9 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 7 \\ -12 \end{pmatrix}$$

has nontrivial solutions. We calculate the the augmented coefficient matrix and see if the last column is pivot column

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 2 & 6 & 3 & 1 \\ 3 & 3 & -3 & 7 \\ 3 & 0 & 0 & -12 \end{array}\right)$$

We perform $S_{12}(-2), S_{13}(-3), S_{14}(-3)$ and get

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & -2 & -1 & -3 \\ 0 & -9 & -9 & 1 \\ 0 & -12 & -6 & -18 \end{array}\right)$$

Then performing $S_{23}(-6)$ see the last column is a free column, so there exist nontrival solutions $(c_1, c_2, c_3) \neq (0,0,0)$, therefore $\mathbf{u} = (2,1,7,-12)^T$ belongs to $\text{Col}(A)$.

1.10 4.(c)

Find the space of all vectors in $\mathbb{R}^4$ that are orthogoanl to $\text{Col}(A)$. Note that the orthogonal complement to $\text{Col}(A)$ is the nullspace of the transpose $\text{Col}(A)^\perp = \text{Null}(A^T)$, therefore we calculate the null space by solving $A^T\mathbf{y} = \mathbf{0}$

$$(A^T|0) = \left(\begin{array}{cccc|c} 1 & 2 & 3 & 3 & 0 \\ 3 & 6 & 9 & 9 & 0 \\ 4 & 6 & 3 & 0 & 0 \\ -1 & 0 & 6 & 9 & 0 \\ 2 & 3 & -3 & 0 & 0 \end{array}\right)$$

We reduce it to RREF by performing $S_{12}(-3), S_{13}(-4), S_{14}(1), S_{15}(-2)$ and get

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -9 & -12 & 0 \\ 0 & 2 & 9 & 12 & 0 \\ 0 & -1 & -9 & -6 & 0 \end{array}\right)$$

Again we perform $S_{34}(11), S_{35}(-1/2), S_5(-2/9), S_3(-1/2)$ we get

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 9/2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

Finally we get

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -9 & 0 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

so $\mathrm{Col(A)}^{\perp} = \mathrm{Null}(A^T) = \{(9,-6,0,1)^T t | t \in \mathbb{R}\}$.

1.11 5.(a)

Let $\mathbf{M}_{2\times2}$ be the vector sapce of all 2×2 matrices with real entries, consider the following subsets of $\mathbb{M}_{2\times2}$

$$U = \left\{ \begin{pmatrix} x & -x \\ y & z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}, W = \left\{ \begin{pmatrix} a & b \\ -a & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

Verify that both U and W are subspaces of $\mathbb{M}_{2\times2}$, and find a basis and the dimensions of U and W. Note that elements of U, W are closed under addition and scalar multiplication, so they are vector subspaces of $\mathbb{M}_{2\times2}$. Since every element in U can be uniquely written as $x(E_{11} - E_{12}) + yE_{21} + E_{22}$ a set of basis of U can be chosen as $\{E_{11} - E_{12}, E_{21}, E_{22}\}$, and U is 3 dimensional. Similarly, a basis for W can be chosen as $\{E_{11} - E_{21}, E_{12}, E_{22}\}$, and W is 3-dimensional.

1.12 5.(b)

Find the dimensions of $U + W, U \cap W$.

Note that $U + W = \mathrm{Span}\{E_{11}, E_{12}, E_{21}, E_{22}\}$ we know $\dim(U + W) = 4$ and has set of basis

$$\{E_{11}, E_{12}, E_{21}, E_{22}\}.$$

The subspace $U \cap W$ has dimension

$$\dim(U \cap W) = \dim U + \dim - \dim(U + W) = 6 - 4 = 2,$$

and $U \cap W$ has basis $\{E_{11} - E_{12} - E_{21}, E_{22}\}$ [1].

1.13 7.(a)

Let $\mathbf{v}_1 = (1,3,0,2)^T, \mathbf{v}_2 = (-1,0,1,0)^T, \mathbf{v}_3 = (5,9,-2,6)^T$ be vectors in $\mathbb{R}^4$. Is it possible to find a set of numbers $\{a_{ij} : 1 \le i, j \le 3\}$ such that the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is liearly independent? Here $\mathbf{w}_i$'s are given by

$$\mathbf{w}_1 = a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + a_{13}\mathbf{v}_3$$

$$\mathbf{w}_2 = a_{21}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + a_{23}\mathbf{v}_3$$

$$\mathbf{w}_3 = a_{31}\mathbf{v}_1 + a_{32}\mathbf{v}_2 + a_{33}\mathbf{v}_3$$

[1] since we know the dimension equals to 2, here one just needs to find a vector different from E_{22} but lies in the intersection.

The condition tells us that $\mathbf{w}_i$ lie in the subspace spanned by $\mathbf{v}_i$. If $\mathbf{v}_i$ are linearly independent, we can take $a_{ij} = \delta_{ij}$. If $\mathbf{v}_i$ are linearly dependent, so the space spanned by $\mathbf{v}_i$ is smaller than 3, then it cannot have three linearly independent vectors. We calculate the maximal linearly independent subset by reduction to RREF

$$\begin{pmatrix} 1 & -1 & 5 \\ 3 & 0 & 9 \\ 0 & 1 & -2 \\ 2 & 0 & 6 \end{pmatrix}$$

We perform $S_{12}(-3), S_{14}(-2)$ to get

$$\begin{pmatrix} 1 & -1 & 5 \\ 0 & 3 & -6 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{pmatrix}$$

Then performing $S_{23}(-1/3), S_{24}(-2/3)$ one see $\dim\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = 2$, therefore such a_{ij}s do not exist.

1.14 7.(b)

You should have already known the fact (from the review problem) that a matrix of the form $A = \mathbf{u}\mathbf{v}^T$ has rank 1, here $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are any n-dimensional non-zero column vectors. What about the converse? That is, is it true that any rank 1 square matrix of size n can be written as $\mathbf{u}\mathbf{v}^T$ for some nonzero column vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$? Prove your claim.

Since A has rank 1, there exists a nonzero column α, since $\dim \mathrm{Col}(A) = \mathrm{rank}(A) = 1$, we know $\mathrm{Col}(A) = \mathrm{Span}(\alpha)$ and therefore the i-th column can be uniquely written as $c_i\alpha$. Then $A = (\gamma)_{n\times 1}(\alpha^T)_{1\times n}$ where $\gamma = (c_1, \dots, c_n)^T$.

1.15

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathbf{w}_1, \mathbf{w}_2$ be two linearly independent sets of vectors in $\mathbb{R}^n$ for some integer n such that $\mathbf{v}_i \cdot \mathbf{w}_j = 0$ for all i, j. Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2\}$ linearly independent? Verify your claim.

Let $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + b_1\mathbf{w}_1 + b_2\mathbf{w}_2 = \mathbf{0}$ be a equation of vectors in $\mathbb{R}^n$, to show $\{\mathbf{v}_i, \mathbf{w}_j\}$ are linearly independent, it suffices to show that $a_i = b_j = 0$. Let us show $\mathbf{b}_i = 0$, the other side is similar. By our condition, taking dot product with $\mathbf{w}_i$ we get $\begin{pmatrix} (\mathbf{w}_1 \cdot \mathbf{w}_1) & (\mathbf{w}_1 \cdot \mathbf{w}_2) \\ (\mathbf{w}_1 \cdot \mathbf{w}_2) & (\mathbf{w}_2 \cdot \mathbf{w}_2) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Note that $\det \begin{pmatrix} (\mathbf{w}_1 \cdot \mathbf{w}_1) & (\mathbf{w}_1 \cdot \mathbf{w}_2) \\ (\mathbf{w}_1 \cdot \mathbf{w}_2) & (\mathbf{w}_2 \cdot \mathbf{w}_2) \end{pmatrix} > ||\mathbf{w}_1||^2 \cdot ||\mathbf{w}_2||^2 - (\mathbf{w}_1 \cdot \mathbf{w}_2)^2 \geq 0$ and equality does not hold because $\mathbf{w}_1$ and $\mathbf{w}_2$ are not proportional. Therefore

$$\begin{pmatrix} (\mathbf{w}_1 \cdot \mathbf{w}_1) & (\mathbf{w}_1 \cdot \mathbf{w}_2) \\ (\mathbf{w}_1 \cdot \mathbf{w}_2) & (\mathbf{w}_2 \cdot \mathbf{w}_2) \end{pmatrix}$$

is invertible and $b_i = 0$. Now the condition reduces to $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$ and we conclude $a_i = 0$ from linear independency of $\mathbf{v}_i$.

Quick solution (By Prof. Xue) Let us write $k_1\mathbf{v} + 1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = -l_1\mathbf{w}_1 - l_2\mathbf{w}_2$, then taking dot product on both sides with $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ one gets $||k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3||^2 = 0$, hence $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = 0$ therefore $k_i = 0$, similarly $l_i = 0$