Student Name: ________________

Student Number: ________________

School: ________________

Year of Entrance: ________________

ShanghaiTech University Midterm Examination Cover Sheet

Academic Year: 2022 to 2023 Term: 1

Course-offering School: IMS

Instructor: Ding, Zhiyuan

Course Name: **Linear Algebra I**

Course Number: MATH1112.05

Exam Instructions for Students:

1. All examination rules must be strictly observed throughout the entire test, and any form of cheating is prohibited.
2. Other than allowable materials, students taking closed-book tests must place their books, notes, tablets and any other electronic devices in places designated by the examiners.
3. Students taking open-book tests may use allowable materials authorized by the examiners. They must complete the exam independently without discussion with each other or exchange of materials.

For Marker's Use:

Section	1	2	3	4	5	6	7	Total
Marks								
Recheck								

Marker's Signature:

Date:

Rechecker's Signature:

Date:

Instructions for Examiners:

1. The format of the exam papers and answer sheets shall be determined by the school and examiners according to actual needs. All pages should be marked by the page numbers in order (except the cover page). All text should be legible, visually comfortable and easy to bind on the left side. A4 double-sided printing is recommended for the convenience of archiving (There are all-in-one printers in the university).

2. The examiners should make sure that exam questions are correct and appropriate, If errors are found in exam questions during the exam, the examiners should be responsible to respond on site, which will be taking into account in the teaching evaluation.

Specific Instructions for students:

- The time duration for this exam is 100 **minutes**.
- Computers and calculators are prohibited in the exam.
- Answers can be written in **either Chinese or English**.

★ For problems 3-7, it is necessary to show details of calculations or deductions. A correct answer with no details can not earn points.

Notations and conventions:

- $\mathbb{R}$ is the set of real numbers.
- I denotes an identity matrix of the suitable order.
- 0 or $\mathbf{0}$ may denote the number zero, a zero vector or a zero matrix.
- $\dim(V)$ is the dimension of a vector space V.
- $\mathsf{tr}(A)$ is the trace of a matrix A.
- $\mathsf{rank}(A)$ is the rank of a matrix A.

1. Fill in the blanks.

(1) (3 points) Let $A = \begin{bmatrix} 4 & 1 & 6 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$. Then $\det(A^{2022}) =$ ____________.

(2) (6 points) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors in $\mathbb{R}^3$ that are orthogonal to one another. Their lenghts are given by $||\mathbf{u}|| = 1, ||\mathbf{v}|| = 2$ and $||\mathbf{w}|| = 3$. Then we have $||\mathbf{u} + \mathbf{v} + \mathbf{w}|| =$ ____________ , $||\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u}|| =$ ____________.

(3) (3 points) Let $\mathbf{u} = [3, -2, 1]^T$ and $\mathbf{v} = [-1, k, -5]^T$ be two vectors in $\mathbb{R}^3$. If the projection of $\mathbf{v}$ along $\mathbf{u}$ has length $= \sqrt{14}$, then $k =$ ____________.

(4) (3 points) Consider two parallel planes in $\mathbb{R}^3$ whose equations are given by $x + 2y + 2z = 4$ and $x + 2y + 2z = -5$. The distance between these two planes is ____________.

(5) (3 points) Let U be a vector space of dimension 6. Let V and W be two subspaces of U such that $\dim(V) = 2$ and $\dim(W) = 3$. Then all possible dimensions of $V + W$ are ____________.

(6) (3 points) Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then $\mathsf{rank}(A) =$ ____________.

2. (10 points) Determine whether the following statements are true or false.

(1) Any system of linear equations with 6 equations and 3 unknowns has no solution.

(2) For any matrix A with 2 rows and 3 columns, the matrix $A^T A$ is always singular.

(3) For any matrix A with 3 rows and 2 columns, the matrix $A^T A$ is always singular.

(4) In any vector space of dimension 5, one can find 4 linearly independent vectors.

(5) Let U be a vector space and let $\mathbf{v}$ and $\mathbf{w}$ be two different vectors in U. Then $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

Write T for true and F for false:

(1)__________ (2)__________ (3)__________ (4)__________ (5)__________

3. (15 points) Let A, B, C be 3×3 matrices such that

$$B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix},$$

and $(I - C^{-1}B)^T C^T A = I$. Find A and A^{-1}.

4. (12 points) Suppose that A and B are 3×3 matrices satisfying

$$A^2B - A - B = I$$

and

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

Compute $\det(B)$.

5. (12 points) Let $A = [a_{ij}]$ be a 4×4 matrix. Assume that the first row of A is given by

$$a_{11} = 1, \quad a_{12} = 2, \quad a_{13} = 0, \quad a_{14} = -4.$$

Let C_{ij} denote the cofactor of A at the i-th row and the j-th column. In other words, C_{ij} is the (i, j)-th entry of the adjoint matrix of A.

Suppose we know that $C_{31} = 6$, $C_{33} = 19$ and $C_{34} = 2$. Compute C_{32}.

6. Consider the matrix

$$A = \begin{bmatrix} 2x & -1 & 2 & 3 \\ x & -2 & 1 & 2 \\ 1 & x & x & x \end{bmatrix}$$

where x is a real number.

(1) (5 points) Prove that $\mathsf{rank}(A) \geq 2$ for all $x \in \mathbb{R}$.

(2) (10 points) Find all $x \in \mathbb{R}$ such that $\mathsf{rank}(A) = 2$.

7. (15 points) Let Z be a vector space of dimension 5. Let U, V, W be subspaces of Z such that

$$\dim(U) = \dim(V) = \dim(W) = 2$$

and

$$U + V + W = Z.$$

Prove that $U \cap V \cap W = \{\mathbf{0}\}$.