Student Name: ____________________

Student Number: ____________________

School/Institute: ____________________

Year of Entrance: ____________________

ShanghaiTech University Midterm Examination

Academic Year: 2021 to 2022 Term: 1

Course-offering School: IMS

Instructor: *Ye, ShuYang* ☐ / *Xue, Boqing* ☐ / *Zheng, Kai* ☐

Course Name: **Linear Algebra I**

Course Number: MATH1112

Exam Instructions for Students:

1. All examination rules must be strictly observed throughout the entire test, and any form of cheating is prohibited.

2. Other than allowable materials, students taking closed-book tests must place their books, notes, tablets and any other electronic devices in places designated by the examiners.

3. Students taking open-book tests may use allowable materials authorized by the examiners. They must complete the exam independently without discussion with each other or exchange of materials.

For Marker's Use:

Problem	1	2	3	4	5	6	7	8	Total
Points	26	8	10	12	12	12	10	10	100
Scores									
Recheck									

Marker's Signature: Rechecker's Signature:

Date: Date:

Specific Instructions:

- Please check the box ☐ behind the name of your instructor on the cover page.
- The time duration for this exam is 120 **minutes**.
- Computers and calculators are prohibit in the exam.
- Answers can be written in **either Chinese or English**.
- You may ask for direct translation during the exam, if needed.

★ For problems 2-8, please show details of calculations or deductions. A correct answer with no details can not earn points.

Notations and conventions:

- $\mathbb{R}$ is the set of real numbers. All the scalars here are real numbers.
- I denotes an identity matrix of suitable order.
- 0 or $\mathbf{0}$ may denote the number zero, a zero vector, or a zero matrix.
- $\det(A)$ is the determinant of a matrix A.
- $\mathsf{tr}(A)$ is the trace of a matrix A.
- $\mathsf{adj}(A)$ is the adjoint (adjunct) matrix of A.
- A linear system is said to be consistent if it has at least one solution.

1. Fill in the blanks.

(1) *(8 points)* In $\mathbb{R}^3$, let $\mathbf{u} = (1, 0, -1)$ and $\mathbf{v} = (0, 1, 2)$.

(a) $\|\mathbf{u} - \mathbf{v}\| =$ ____________ ;

(b) $\mathbf{u} \cdot \mathbf{v} =$ ____________ ;

(c) $\mathbf{u} \times \mathbf{v} =$ ____________;

(d) The orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$ is $\text{proj}_{\mathbf{v}}(\mathbf{u}) =$ ____________

(2) *(4 points)* Let $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$ and $p(x) = (x+1)(x-1)$. Then

$$p(A) = \_\_\_\_\_\_\_\_\_\_\_\_.$$

(3) *(4 points)* The inverse of the matrix $A = \begin{bmatrix} -2 & 1 \\ 5 & 3 \end{bmatrix}$ is ____________

(4) *(6 points)* Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 6 \end{bmatrix}$. Compute the following trace and determinant:

$$\text{tr}(A^T) = \_\_\_\_\_\_\_\_\_\_\_\_, \quad \det\left(A^{2021}\right) = \_\_\_\_\_\_\_\_\_\_\_\_.$$

(5) *(4 points)* In $\mathbb{R}^3$, suppose that Π is the plane given by $2x - y - 3z + 2 = 0$. The distance between the point $P_0(1, -1, 0)$ and the plane Π is ____________.

2. *(8 points)* Find an invertible matrix P such that $PA = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} - a_{21} & a_{32} - a_{22} & a_{33} - a_{23} \end{bmatrix}.$$

3. *(10 points)* Let $c \in \mathbb{R}$ be a parameter. Suppose that

$$p_1(x) = 1 - 2x, \quad p_2(x) = 3 + x - cx^2, \; p_3(x) = -1 + 3x^2,$$

$$p_4(x) = 1 + 2021x + 2021^2x^2 + 2021^3x^3.$$

Find a value of c such that p_1, p_2, p_3, p_4 are *linearly dependent* in the vector space P_∞ of all polynomials.

4. *A square matrix* A *is called idempotent if* $A^2 = A$.

A square matrix A *is said to be involutory if* $A^2 = I$.

(1) *(6 points)* Suppose that A, B are both idempotent. Prove that $A + B$ is idempotent if and only if $AB + BA = 0$.

(2) *(6 points)* Suppose that A, B are both involutory. Prove that AB is involutory if and only if $AB = BA$.

5. Let A be an $n \times n$ invertible matrix. Prove the following two statements:

(1) *(6 points)* $\mathsf{adj}(A^{-1}) = \big(\mathsf{adj}(A)\big)^{-1}$.

(2) *(6 points)* $\mathsf{adj}\big(\mathsf{adj}(A)\big) = \big(\det(A)\big)^{n-2} A$.

6. *(12 points)* Let $a, b \in \mathbb{R}$ be parameters. Consider the linear system of equations

$$\begin{cases} -2x + y + z = -2 \\ x - 2y + z = a \\ x + y + (b-2)z = a^2 + b \end{cases}$$

When is the above linear system consistent? When it is consistent, find the solutions.

7. *(10 points)* Evaluate the following determinant of order n ($n \geq 2$):

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 1 & 2 & \dots & n-2 & n-1 \\ 3 & 2 & 1 & \dots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-2 & n-3 & \dots & 1 & 2 \\ n & n-1 & n-2 & \dots & 2 & 1 \end{vmatrix}.$$

8. *A matrix A is said to be skew-symmetric if $A^T = -A$.*

Let A be an $n \times n$ skew-symmetric matrix with $n \geq 2$.

(1) *(2 points)* Suppose that n is odd (i.e., $n = 2k - 1$). Prove that $\det(A) = 0$.

(2) *(8 points)* Suppose that n is even (i.e., $n = 2k$) and $\det(A) = 0$. Prove that the linear system $A\mathbf{x} = \mathbf{0}$ has at least two linearly independent solutions.