Linear Algebra: Final Exam Questions

ShanghaiTech University

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Name: ______________________________

Student ID: _________________________

Lecturer: ___________________________

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Problem 1. (10 pts) Let A be the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the general expression of A^n for any positive integer n.

Problem 2. (16 pts) Consider a matrix

$$A = \begin{bmatrix} 3 & 0 & 5 & 2 & 1 \\ -3 & -12 & -4 & 5 & -5 \\ 1 & -4 & 2 & 3 & -1 \end{bmatrix}.$$

(1) (4 pts) Find the reduced row echelon form of A.
(2) (4 pts) Give the general solution of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.
(3) (4 pts) Does *any* non-homogeneous linear system in the form of $A\mathbf{x} = \mathbf{b}$ with the given coefficient matrix A and a nonzero column vector $\mathbf{b} = [b_1 \ b_2 \ b_3]^T$ have a solution? If you think this is true, explain the reason; otherwise provide a counterexample.
(4) (4 pts) Find the dimension of the row space of A, column space of A, null space of A and null space of A^T. Here the row space of A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A, and we define the column space similarly.

Problem 3. (12 pts) Let $P(1,2,3)$ be a point in $\mathbb{R}^3$ whose coordinates are $(x,y,z) = (1,2,3)$. Let H be a plane in $\mathbb{R}^3$ defined by the equation $2x - 6y + z = 0$.

(1) (2 pts) Find the normal vector $\mathbf{n}$ of H with $\|\mathbf{n}\| = 1$.
(2) (4 pts) Find the distance between $P(1,2,3)$ and H.
(3) (6 pts) Let $Q(3,1,0)$ be another point in $\mathbb{R}^3$ whose coordinates are $(x,y,z) = (3,1,0)$. Let L be a line that is contained in H and orthogonal to $\overrightarrow{OP} = (1,2,3)$ and passes through $Q(3,1,0)$. Find the equation for L in the parametric form $\mathbf{x}_0 + t\mathbf{v}$, with a parameter t and appropriate vectors $\mathbf{x}_0$ and $\mathbf{v}$.

Problem 4. (12 pts) Consider the quadratic form

$$q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3.$$

(1) (2 pts) Find a symmetric matrix A such that $q(x) = x^T Ax$.
(2) (4 pts) Find the eigenvalues λ_1, λ_2 and λ_3 of the matrix A, and put them in decreasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3$.
(3) (6 pts) Find an orthogonal change of variable $x = Py$ such that $q(x)$ can be expressed as

$$q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2.$$

Problem 5. (10 pts) Consider the vector space P_n of polynomials of degree at most n.

(1) (4 pts) Prove that

$$\langle p, q\rangle = \int_0^1 p'(x)q'(x)\,dx + p(0)q(0)$$

is an inner product on this space, where $p'(x)$ denotes the derivative of $p(x)$. (You may assume that $\langle P, Q\rangle = \int_0^1 P(x)Q(x)\,dx$ defines an inner product on P_k for all k.)

(2) (6 pts) Let $n = 3$. Find a basis for P_3 that is orthonormal with respect to this inner product.

Problem 6. (16 pts) Let M_{22} be the vector space of 2×2 matrices with real entries. Define the inner product $\langle A, B \rangle = \mathrm{tr}(AB^T)$ on M_{22}, where B^T is the transpose of B and tr denotes the trace of a square matrix. Define $T : M_{22} \to M_{22}$ by

$$T(X) = X + X^T.$$

(1) (4 pts) Show that T is a linear transformation.

(2) (4 pts) Find the matrix for T relative to the basis

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

(3) (4 pts) Find the eigenvalues and bases for the eigenspaces of T.

(4) (4 pts) Find an orthonormal basis B' of M_{22} such that the matrix for T relative to B' is diagonal.

Problem 7. (12 pts) Define $a_0 := 1, a_1 := 1$, and inductively for each integer $n \geq 0$, $a_{n+2} := a_{n+1} + a_n$.

(*a*) (6 pts) Prove that we have, for each integer $n \geq 0$, $\begin{pmatrix} a_{n+2} & a_{n+1} \\ a_{n+1} & a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+1}$.

(*b*) (6 pts) Use (*a*) to calculate a_{1000} in terms of $\beta = \frac{1+\sqrt{5}}{2}$.

Problem 8. (12 pts) Let A be an $n \times n$ matrix with all distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ $(r \leq n)$. Use the theory in this course to prove:

(1) (4 pts) If A is diagonalizable, then

$$(\lambda_1 I - A)(\lambda_2 I - A) \cdots (\lambda_r I - A) = 0. \qquad (*)$$

(2) (8 pts) For any positive integer n, the condition $(*)$ implies that A is diagonalizable.

If you don't know how to prove (2), you can try to do the following:

(3) (4 pts) If $n = 3$ and $(*)$ holds, then A is diagonalizable.