第四章，第五章复习自测题

1 不定项选择, Multiple Choices. 15 points

1-1 (5 points). Let V be a vector space. Determine which of the following statements are true.

- (A) Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset V$ $(r \geq 2)$. If $\boldsymbol{v}_1$ cannot be expressed as a linear combination of $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$, then the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is always linearly independent.
- (B) Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset V$ $(r \geq 2)$. If there is another subset $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r-1}\} \subset V$ such that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset \operatorname{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r-1}\}$, then the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is always linearly dependent.
- (C) Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset V$. If there is another subset $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\} \subset V$ for some integer $m \geq 1$ such that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset \operatorname{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$, then the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is always linearly dependent.
- (D) For $\boldsymbol{v}, \boldsymbol{w}_1, \boldsymbol{w}_2 \in V$, if $\boldsymbol{v}$ and $\boldsymbol{w}_1$ are linearly independent, $\boldsymbol{v}$ and $\boldsymbol{w}_2$ are linearly independent, $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are linearly independent, then the set $\{\boldsymbol{v}, \boldsymbol{w}_1, \boldsymbol{w}_2\}$ is always also linearly independent.

1-2 (5 points). Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset \mathbb{R}^n$ be a subset of vectors in $\mathbb{R}^n$, $1 \leq r \leq n$. Determine which of the following statements are true.

- (A) If there is a linearly independent subset $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r\} \subset \mathbb{R}^n$ such that $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r\} \subset \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$, then the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is always linearly independent.
- (B) We consdier $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ as $n \times 1$-column vector, and define $\boldsymbol{w}_1 = \begin{bmatrix} \boldsymbol{v}_1 \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$, $\boldsymbol{w}_2 = \begin{bmatrix} \boldsymbol{v}_2 \\ 2 \end{bmatrix} \in \mathbb{R}^{n+1}$, ..., $\boldsymbol{w}_r = \begin{bmatrix} \boldsymbol{v}_r \\ r \end{bmatrix} \in \mathbb{R}^{n+1}$ (For example, if $n = 2$ and $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \in \mathbb{R}^2$, then $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \in \mathbb{R}^3$). If $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r\} \subset \mathbb{R}^{n+1}$ is linearly independent, then the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is always linearly independent.
- (C) Let $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r\} \subset \mathbb{R}^n$ be another subset of vectors in $\mathbb{R}^n$, then there always exists a linear transformation $T : \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \to \mathbb{R}^n$ such that $T(\boldsymbol{v}_1) = \boldsymbol{w}_1, \ldots, T(\boldsymbol{v}_r) = \boldsymbol{w}_r$.
- (D) If the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset \mathbb{R}^n$ is linearly independent, then for any given subset

$\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r\} \subset \mathbb{R}^n$, there always exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\boldsymbol{v}_1) = \boldsymbol{w}_1, \ldots, T(\boldsymbol{v}_r) = \boldsymbol{w}_r$.

1-3 (5 points). Determine which of the following statements are true.

- (A) For any $A \in M_{m\times n}$ and $B \in M_{m\times n}$, we always have $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.
- (B) For any $A \in M_{n\times n}$ and $B \in M_{n\times n}$, if A and B are similar, then $A^\top$ and $B^\top$ are also similar.
- (C) For any$A \in M_{n\times n}$ and $B \in M_{n\times n}$, if A^2 and B^2 are similar, then A and B are also similar.
- (D) Let $A, B, C \in M_{n\times n}$ be such that $AB = C$ and B is invertible. Then $\text{rank}(A) = \text{rank}(C)$.

2 填空题, Fill in the blanks. 15 points

2-1 (5 points). Let $V = \text{span}\{e^x, e^{-x}\} \subset C(-\infty, \infty)$ be a subspace of the vector space of all continuous functions on $\mathbb{R}$. The hyperbolic functions are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Let $B = \{e^x, e^{-x}\}$ and $B' = \{\sinh(x), \cosh(x)\}$. Then the transition matrix $P_{B'\leftarrow B}$ from B to B' is equal to ______.

2-2 (5 points). Let

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

We can show that the set $S = \{A_1, A_2, A_3, A_4\}$ is a basis of $M_{2\times 2}$. Then the coordinate vector of the matrix $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ relative to S is $[A]_S =$ ______.

2-3 (5 points). Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Let $P_\infty = \{p(x) = a_0 + a_1 x + \ldots + a_n x^n : n \geq 0, a_0, \ldots, a_n \in \mathbb{R}\}$ be the vector space of all polynomials. Define

$$V = \text{span}\{p(A) = a_0 I_2 + a_1 A + \ldots + a_n A^n : p(x) \in P_\infty\} \subset M_{2\times 2}.$$

Then $\dim(V) =$ ______.

3 10 points

Consider the bases $B = \{p_1(x), p_2(X)\}$ and $B' = \{q_1(x), q_2(x)\}$ for $P_1 = \{a_0 + a_1x : a_0, a_1 \in \mathbb{R}\}$, where

$$p_1(x) = 6 + 3x, p_2(x) = 10 + 2x, q_1(x) = 2, q_2(x) = 3 + 2x.$$

1. (3 points) Find the transition matrix $P_{B \leftarrow B'}$ from B' to B.
2. (3 points)Find the transition matrix $P_{B' \leftarrow B}$ from B to B'.
3. (4 points) For $h(x) = -4 + x$, find its coordinate vectors $[h]_B$ and $[h]_{B'}$.

4 10 points

Let $P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ and $P_2 = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$. Let $T_1 : P_3 \to P_2$ be a linear transformation defined by

$$T_1(p(x)) = p'(x) + p''(x)$$

(here $p'(x)$ denotes the derivative of $p(x)$ and $p''(x)$ denotes the derivative of $p'(x)$), and $T_2 : P_2 \to P_3$ be a linear transformation defined by

$$T_2(p(x)) = xp(2x + 1).$$

Let $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$ be the standard basis of P_2 and P_3 respectively.

1. (4 points) Find the expression of $(T_1 \circ T_2)(p(x))$ for $p(x) = a_0 + a_1x + a_2x^2 \in P_2$.
2. (6 points) Find the matrix $[T_2 \circ T_1]_{B',B'}$ by using the product of $[T_2]_{B',B}$ and $[T_1]_{B,B'}$.

5 10 points

Consider the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose standard matrix is

$$[T] = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

Let $B' = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ be another basis of $\mathbb{R}^2$, where $\boldsymbol{u}_1 = (1, 1), \boldsymbol{u}_2 = (1, 2)$. Find the matrix of T relative to the basis B', i.e., $[T]_{B',B'}$, and show its relation with the standard matrix $[T]$.

6 10 points

In $\mathbb{R}^3$, let

$$\boldsymbol{x}_1 = (1, -2, -5), \boldsymbol{x}_2 = (0, 8, 9),$$

and

$$\boldsymbol{y}_1 = (1, 6, 4), \boldsymbol{y}_2 = (2, 4, -1), \boldsymbol{y}_3 = (-1, 2, 5).$$

Determine whether span$\{\boldsymbol{x}_1, \boldsymbol{x}_2\} =$ span$\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$, and explain your argument.

7 15 points

Let V be a finite dimensional vector space. Let $T : V \to V$ be a linear operator satisfying $T^3 = T \circ T \circ T = 4T$. Prove that $\ker(T) + \text{RAN}(T^2) = V$, here $\ker(T)$ denotes the kernel of T, RAN(T) denotes the range of T.

8 15 points

Let $A \in M_{3\times 3}$ be a matrix such that its first row is nonzero, i.e., if A is expressed by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then $(a_{11}, a_{12}, a_{13}) \neq (0, 0, 0)$. Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \end{bmatrix}$. Assume that $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
Compute the nullity of A, and find a basis of Null(A).