第四章，第五章复习自测题

1 不定项选择, Multiple Choices. 15 points

1-1 (5 points). Let V be a vector space. Determine which of the following statements are true.

- (A) Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset V$ $(r \geq 2)$. If $\boldsymbol{v}_1$ cannot be expressed as a linear combination of $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$, then the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is always linearly independent.

  错误。反例：令$V = \mathbb{R}^2$，$\boldsymbol{v}_1 = (1, 0)$，$\boldsymbol{v}_2 = (0, 1)$，$\boldsymbol{v}_3 = (0, 2)$。显然$\boldsymbol{v}_1$不能写成$\boldsymbol{v}_2$与$\boldsymbol{v}_3$的线性组合，但是由于$\boldsymbol{v}_3 = 2\boldsymbol{v}_2$，集合$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$线性相关。

- (B) Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset V$ $(r \geq 2)$. If there is another subset $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r-1}\} \subset V$ such that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset \text{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r-1}\}$, then the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is always linearly dependent.

  正确。由假设$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset \text{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r-1}\}$可知$\dim(\text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}) \leq \dim(\text{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r-1}\}) = r-1$。所以如果$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$线性无关，那么必然会有$\dim(\text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}) = r > \dim(\text{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r-1}\}) = r-1$，矛盾！因此$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$必然线性相关。

- (C) Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset V$. If there is another subset $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\} \subset V$ for some integer $m \geq 1$ such that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset \text{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$, then the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is always linearly dependent.

  错误。反例：令$V = \mathbb{R}^3$，令$\boldsymbol{v}_1 = \boldsymbol{w}_1 = (1, 0, 0)$，$\boldsymbol{v}_2 = \boldsymbol{w}_2 = (0, 1, 0)$，$\boldsymbol{w}_3 = (0, 0, 1)$，那么显然$\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \subset \text{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$，但是$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$线性无关。

- (D) For $\boldsymbol{v}, \boldsymbol{w}_1, \boldsymbol{w}_2 \in V$, if $\boldsymbol{v}$ and $\boldsymbol{w}_1$ are linearly independent, $\boldsymbol{v}$ and $\boldsymbol{w}_2$ are linearly independent, $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are linearly independent, then the set $\{\boldsymbol{v}, \boldsymbol{w}_1, \boldsymbol{w}_2\}$ is always also linearly independent.

  错误。反例：令$V = \mathbb{R}^3$，$\boldsymbol{v} = (1, 0, 0)$，$\boldsymbol{w}_1 = (0, 1, 0)$，$\boldsymbol{w}_2 = (1, 1, 0)$。容易验证$\boldsymbol{v}$与$\boldsymbol{w}_1$线性无关，$\boldsymbol{v}$与$\boldsymbol{w}_2$线性无关，$\boldsymbol{w}_1$与$\boldsymbol{w}_2$线性无关，但是由于$\boldsymbol{w}_2 =$

$\boldsymbol{v}+\boldsymbol{w}_1$，集合$\{\boldsymbol{v},\boldsymbol{w}_1,\boldsymbol{w}_2\}$线性相关。

这道题目告诉我们，**对于一个集合$\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}$，它里面的向量两两线性无关不能保证整个集合是线性无关的。**

1-2 (5 points). Let $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}\subset\mathbb{R}^n$ be a subset of vectors in $\mathbb{R}^n$, $1\le r\le n$. Determine which of the following statements are true.

- (A) If there is a linearly independent subset $\{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_r\}\subset\mathbb{R}^n$ such that $\{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_r\}\subset\text{span}\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}$, then the set $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}$ is always linearly independent.

  正确。$\{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_r\}$线性无关意味着$\dim(\text{span}\{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_r\})=r$，因此由$\{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_r\}\subset\text{span}\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}$可得$\dim(\text{span}\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\})\ge r$。如果$\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}$线性相关，那么必然会有$\dim(\text{span}\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\})<r$，矛盾！故$\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}$线性无关。

- (B) We consdier $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r$ as $n\times 1$-column vector, and define $\boldsymbol{w}_1=\begin{bmatrix}\boldsymbol{v}_1\\1\end{bmatrix}\in\mathbb{R}^{n+1}$, $\boldsymbol{w}_2=\begin{bmatrix}\boldsymbol{v}_2\\2\end{bmatrix}\in\mathbb{R}^{n+1}$, …, $\boldsymbol{w}_r=\begin{bmatrix}\boldsymbol{v}_r\\r\end{bmatrix}\in\mathbb{R}^{n+1}$ (For example, if $n=2$ and $\boldsymbol{v}_1=\begin{bmatrix}1\\3\end{bmatrix}\in\mathbb{R}^2$, then $\boldsymbol{w}_1=\begin{bmatrix}1\\3\\1\end{bmatrix}\in\mathbb{R}^3$). If $\{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_r\}\subset\mathbb{R}^{n+1}$ is linearly independent, then the set $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}$ is always linearly independent.

  错误。 令$r=2$，$n=2$，$\boldsymbol{v}_1=\begin{bmatrix}1\\0\end{bmatrix}=\boldsymbol{v}_2$，那么必然有$\boldsymbol{v}_1$与$\boldsymbol{v}_2$线性相关，但$\boldsymbol{w}_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$与$\boldsymbol{w}_2=\begin{bmatrix}1\\0\\2\end{bmatrix}$线性无关。

- (C) Let $\{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_r\}\subset\mathbb{R}^n$ be another subset of vectors in $\mathbb{R}^n$, then there always exists a linear transformation $T:\text{span}\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}\to\mathbb{R}^n$ such that $T(\boldsymbol{v}_1)=\boldsymbol{w}_1,\ldots,T(\boldsymbol{v}_r)=\boldsymbol{w}_r$.

  错误。 令$r=2$，假设$\boldsymbol{v}_2=2\boldsymbol{v}_1$，$\boldsymbol{v}_1\neq\mathbf{0}$，但$\boldsymbol{w}_2=3\boldsymbol{w}_1$，$\boldsymbol{w}_1\neq\mathbf{0}$。如果存在一个线性变换$T:\mathbb{R}^n\to\mathbb{R}^n$使得$T(\boldsymbol{v}_1)=\boldsymbol{w}_1$，$T(\boldsymbol{v}_2)=\boldsymbol{w}_2$，那么$T$的线性性质必须使得$\boldsymbol{w}_2=T(\boldsymbol{v}_2)=T(2\boldsymbol{v}_1)=2T(\boldsymbol{v}_1)=2\boldsymbol{w}_1$，这与假设$\boldsymbol{w}_2=3\boldsymbol{w}_1$，$\boldsymbol{w}_1\neq\mathbf{0}$矛盾。因此这种情况下不存在这样的线性变换$T$。

- (D) If the set $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}\subset\mathbb{R}^n$ is linearly independent, then for any given subset

$\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r\} \subset \mathbb{R}^n$, there always exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\boldsymbol{v}_1) = \boldsymbol{w}_1, \ldots, T(\boldsymbol{v}_r) = \boldsymbol{w}_r$.

正确。 如果$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$线性无关，那么可以将其扩充为$\mathbb{R}^n$的一组基底

$$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r, \boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_n\}$$

那么任何$\boldsymbol{v} \in \mathbb{R}^n$可以**唯一**表示为$\boldsymbol{v} = k_1\boldsymbol{v}_1 + \ldots + k_n\boldsymbol{v}_n$。我们定义$T : \mathbb{R}^n \to \mathbb{R}^n$为

$$T(\boldsymbol{v}) = T(k_1\boldsymbol{v}_1 + \ldots + k_n\boldsymbol{v}_n) = k_1\boldsymbol{w}_1 + \ldots + k_r\boldsymbol{w}_r + k_{r+1}\mathbf{0} + \ldots + k_n\mathbf{0}.$$

显然这样的T是一个线性变换(请自己验证一下)，且满足$T(\boldsymbol{v}_1) = \boldsymbol{w}_1, \ldots, T(\boldsymbol{v}_r) = \boldsymbol{w}_r$。

1-3 (5 points). Determine which of the following statements are true.

- (A) For any $A \in M_{m\times n}$ and $B \in M_{m\times n}$, we always have $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

  正确。 将A与B写为列向量表示

  $$A = \begin{bmatrix}\boldsymbol{v}_1 & \ldots & \boldsymbol{v}_n\end{bmatrix}, B = \begin{bmatrix}\boldsymbol{w}_1 & \ldots & \boldsymbol{w}_n\end{bmatrix},$$

  那么$A + B = \begin{bmatrix}\boldsymbol{v}_1 + \boldsymbol{w}_1 & \ldots & \boldsymbol{v}_n + \boldsymbol{w}_n\end{bmatrix}$。因此$A + B$的列空间$\text{Col}(A + B) = \text{span}\{\boldsymbol{v}_1+\boldsymbol{w}_1, \ldots, \boldsymbol{v}_n+\boldsymbol{w}_n\} \subset \text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}+\text{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\} = \text{Col}(A)+\text{Col}(B)$。因此显然有

  $$\dim(\text{Col}(A + B)) \leq \dim(\text{Col}(A) + \text{Col}(B)) \leq \dim(\text{Col}(A)) + \dim(\text{Col}(B)).$$

- (B) For any $A \in M_{n\times n}$ and $B \in M_{n\times n}$, if A and B are similar, then $A^\top$ and $B^\top$ are also similar.

  正确。 如果A与B相似，那么存在可逆矩阵P使得

  $$B = P^{-1}AP,$$

  那么两边同时转置，可得$B^\top = (P^{-1}AP)^\top = P^\top A^\top (P^{-1})^\top = P^\top A^\top (P^\top)^{-1} = Q^{-1}A^\top Q$，$Q = (P^\top)^{-1}$。因此$A^\top$与$B^\top$相似。

- (C) For any$A \in M_{n\times n}$ and $B \in M_{n\times n}$, if A^2 and B^2 are similar, then A and B are also similar.

错误。 令$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$，$B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$。显然$\mathrm{rank}(A) = 0 \neq 1 = \mathrm{rank}(B)$，因此$A$与$B$不相似(如果他们相似，那么由于矩阵的秩是相似不变量，则有$\mathrm{rank}(A) = \mathrm{rank}(B)$，与事实不符)。但是$A^2 = B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$，即$A^2$与$B^2$相似。

- (D) Let $A, B, C \in M_{n\times n}$ be such that $AB = C$ and B is invertible. Then $\mathrm{rank}(A) = \mathrm{rank}(C)$.

正确。 第九次作业Problem E告诉我们，对于可逆矩阵B诱导的矩阵变换T_B，即$T_B : \mathbb{R}^n \to \mathbb{R}^n$为同构，我们有

$$\mathrm{rank}(T_C) = \mathrm{rank}(T_A \circ T_B) = \mathrm{rank}(T_A).$$

因此有$\mathrm{rank}(A) = \mathrm{rank}(C)$.

2 填空题, Fill in the blanks. 15 points

2-1 (5 points). Let $V = \mathrm{span}\{e^x, e^{-x}\} \subset C(-\infty, \infty)$ be a subspace of the vector space of all continuous functions on $\mathbb{R}$. The hyperbolic functions are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Let $B = \{e^x, e^{-x}\}$ and $B' = \{\sinh(x), \cosh(x)\}$. Then the transition matrix $P_{B' \leftarrow B}$ from B to B' is equal to ______.

答案： 由讲义116页关于转移矩阵的定义可知，

$$P_{B'\leftarrow B} = \begin{bmatrix} [e^x]_{B'} & [e^{-x}]_{B'} \end{bmatrix},$$

由于$e^x = \sinh(x) + \cosh(x)$，我们有$[e^x]_{B'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$，又由$e^{-x} = \cosh(x) - \sinh(x)$可得$[e^{-x}]_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$，代入以上$P_{B'\leftarrow B}$的表达式可得

$$P_{B'\leftarrow B} = \begin{bmatrix} [e^x]_{B'} & [e^{-x}]_{B'} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

2-2 (5 points). Let

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

We can show that the set $S=\{A_1,A_2,A_3,A_4\}$ is a basis of $M_{2\times 2}$. Then the coordinate vector of the matrix $A=\begin{bmatrix}2&0\\-1&3\end{bmatrix}$ relative to S is $[A]_S=$ ______.

答案： 假设$[A]_S=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$，那么有

$$\begin{aligned}A=\begin{bmatrix}2&0\\-1&3\end{bmatrix}&=x_1A_1+x_2A_2+x_3A_3+x_4A_4\\&=x_1\begin{bmatrix}-1&1\\0&0\end{bmatrix}+x_2\begin{bmatrix}1&1\\0&0\end{bmatrix}+x_3\begin{bmatrix}0&0\\1&1\end{bmatrix}+x_4\begin{bmatrix}0&-1\\0&1\end{bmatrix}\\&=\begin{bmatrix}-x_1+x_2&x_1+x_2-x_4\\x_3&x_3+x_4\end{bmatrix},\end{aligned}$$

即x_1,x_2,x_3,x_4必须满足方程组

$$\begin{aligned}-x_1+x_2&=2\\x_1+x_2-x_4&=0\\x_3&=-1\\x_3+x_4&=3.\end{aligned}$$

求解该方程组可得

$$x_1=1,x_2=3,x_3=-1,x_4=4$$

即$[A]_S=\begin{bmatrix}1\\3\\-1\\4\end{bmatrix}$。

2-3 (5 points). Let $A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$. Let $P_\infty=\{p(x)=a_0+a_1x+\ldots+a_nx^n : n\geq 0, a_0,\ldots,a_n\in\mathbb{R}\}$ be the vector space of all polynomials. Define

$$V=\text{span}\{p(A)=a_0I_2+a_1A+\ldots+a_nA^n : p(x)\in P_\infty\}\subset M_{2\times 2}.$$

Then $\dim(V)=$ ______.

答案： 由于$V=\text{span}\{p(A)=a_0I_2+a_1A+\ldots+a_nA^n : p(x)\in P_\infty\}$，可知$V$的任何向量$\boldsymbol{v}\in V$都可以表示为

$$\boldsymbol{v}=k_1p_1(A)+\ldots+k_mp_m(A),$$

这里$m \geq 1, k_1, \ldots, k_m \in \mathbb{R}, p_1(x), \ldots, p_m(x) \in P_\infty$。特别的，任何$\boldsymbol{v} \in V$都可以表示为$I_2, A, A^2, \ldots, A^n$的线性组合，这里$n$可以取任何自然数。

接下来我们注意到，因为$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$，我们有$A^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I_2$，由此可得$A^3 = A^2A = -I_2A = -A$，$A^4 = A^3A = -AA = -A^2 = I_n, \ldots$，因此实际上$V = \text{span}\{I_2, A\}$。容易验证$I_2$与$A$线性无关，故$\dim(V) = 2$。

3 10 points

Consider the bases $B = \{p_1(x), p_2(x)\}$ and $B' = \{q_1(x), q_2(x)\}$ for $P_1 = \{a_0 + a_1x : a_0, a_1 \in \mathbb{R}\}$, where

$$p_1(x) = 6 + 3x, p_2(x) = 10 + 2x, q_1(x) = 2, q_2(x) = 3 + 2x.$$

1. (3 points) Find the transition matrix $P_{B \leftarrow B'}$ from B' to B.

方法1： 令$S = \{1, x\}$为P_1的标准基底。那么

$$\tilde{B} = \{[p_1(x)]_S = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, [p_2(x)]_S = \begin{bmatrix} 10 \\ 2 \end{bmatrix}\}$$

与

$$\tilde{B}' = \{[q_1(x)]_S = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, [q_2(x)]_S = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\}$$

是$\mathbb{R}^2$的两组基底。我们不妨用$\tilde{B}$指代矩阵$\begin{bmatrix} 6 & 10 \\ 3 & 2 \end{bmatrix}$，用$\tilde{B}'$来指代矩阵$\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$，那么利用讲义117-118页的算法，用初等行变换同时作用在$[\tilde{B}|\tilde{B}']$上直到左边的$\tilde{B}$变为单位矩阵，此时右边的矩阵$\tilde{B}'$变为转移矩阵$P_{\tilde{B} \leftarrow \tilde{B}'}$，也即$P_{\tilde{B} \leftarrow \tilde{B}'} = (\tilde{B})^{-1}\tilde{B}'$。经过具体计算可得$P_{\tilde{B} \leftarrow \tilde{B}'} = (\tilde{B})^{-1}\tilde{B}' = \begin{bmatrix} -2/9 & 7/9 \\ 1/3 & -1/6 \end{bmatrix}$。因此得到$P_{B \leftarrow B'} = P_{\tilde{B} \leftarrow \tilde{B}'} = \begin{bmatrix} -2/9 & 7/9 \\ 1/3 & -1/6 \end{bmatrix}$。

方法2： 由讲义116页关于转移矩阵的定义可知，

$$P_{B \leftarrow B'} = \begin{bmatrix} [q_1(x)]_B & [q_2(x)]_B \end{bmatrix},$$

因此只需确定两个列向量$[q_1(x)]_B$与$[q_2(x)]_B$。假设$[q_1(x)]_B = \begin{bmatrix} a \\ b \end{bmatrix}$，那么它需要满足

$$q_1(x) = ap_1(x) + bp_2(x) = a(6 + 3x) + b(10 + 2x) = (6a + 10b) + (3a + 2b)x,$$

代入$q_1(x)=2$，可得$6a+10b=2, 3a+2b=0$。求解该方程组，可得$a=-2/9, b=1/3$，即

$$[q_1(x)]_B = \begin{bmatrix} -2/9 \\ 1/3 \end{bmatrix}.$$

类似地，假设$[q_1(x)]_B = \begin{bmatrix} c \\ d \end{bmatrix}$，那么它需要满足

$$q_2(x) = cp_1(x) + dp_2(x) = c(6+3x) + d(10+2x) = (6c+10d) + (3c+2d)x,$$

代入$q_2(x)=3+2x$，可得$6c+10d=3, 3c+2d=2$。求解该方程组，可得$c=7/9, d=-1/6$，即

$$[q_2(x)]_B = \begin{bmatrix} 7/9 \\ -1/6 \end{bmatrix}.$$

因此，

$$P_{B\leftarrow B'} = \begin{bmatrix} [q_1(x)]_B & [q_2(x)]_B \end{bmatrix} = \begin{bmatrix} -2/9 & 7/9 \\ 1/3 & -1/6 \end{bmatrix}.$$

2. (3 points)Find the transition matrix $P_{B'\leftarrow B}$ from B to B'.

   答案： $P_{B'\leftarrow B} = P_{B\leftarrow B'}^{-1} = \begin{bmatrix} 3/4 & 7/2 \\ 3/2 & 1 \end{bmatrix}$。

3. (4 points) For $h(x) = -4 + x$, find its coordinate vectors $[h]_B$ and $[h]_{B'}$.

   答案： 令$[h(x)]_B = \begin{bmatrix} a \\ b \end{bmatrix}$，那么$h(x) = -4 + x = ap_1(x) + bp_2(x) = a(6 + 3x) + b(10 + 2x) = (6a + 10b) + (3a + 2b)x$，即$6a + 10b = -4, 3a + 2b = 1$。求解该方程组可得$a = 1, b = -1$，即$[h(x)]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$，由此可得$[h(x)]_{B'} = P_{B'\leftarrow B}[h(x)]_B = \begin{bmatrix} 3/4 & 7/2 \\ 3/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -11/4 \\ 1/2 \end{bmatrix}$。

4 10 points

Let $P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ and $P_2 = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$. Let $T_1 : P_3 \to P_2$ be a linear transformation defined by

$$T_1(p(x)) = p'(x) + p''(x)$$

(here $p'(x)$ denotes the derivative of $p(x)$ and $p''(x)$ denotes the derivative of $p'(x)$), and $T_2 : P_2 \to P_3$ be a linear transformation defined by

$$T_2(p(x)) = xp(2x + 1).$$

Let $B=\{1,x,x^2\}$ and $B'=\{1,x,x^2,x^3\}$ be the standard basis of P_2 and P_3 respectively.

1. (4 points) Find the expression of $(T_1\circ T_2)(p(x))$ for $p(x)=a_0+a_1x+a_2x^2\in P_2$.

   答案： 由于$T_2(1)=x,T_1(x)=1$，可得$(T_1\circ T_2)(1)=1$；由于$T_2(x)=x(2x+1)=2x^2+x$，$T_1(2x^2+x)=(2x^2+x)'+(2x^2+x)''=4x+5$，可得$(T_1\circ T_2)(x)=4x+5$；由于$T_2(x^2)=x(2x+1)^2=4x^3+4x^2+x$，$T_1(4x^3+4x^2+x)=(4x^3+4x^2+x)'+(4x^3+4x^2+x)''=12x^2+32x+9$，可得$(T_1\circ T_2)(x^2)=12x^2+32x+9$。因此

$$\begin{aligned}(T_1\circ T_2)(p(x))&=(T_1\circ T_2)(a_0+a_1x+a_2x^2)\\&=a_0(T_1\circ T_2)(1)+a_1(T_1\circ T_2)(x)+a_2(T_1\circ T_2)(x^2)\\&=a_0\times 1+a_1(4x+5)+a_2(12x^2+32x+9)\\&=(a_0+5a_1+9a_2)+(4a_1+32a_2)x+(12a_2)x^2.\end{aligned}$$

2. (6 points) Find the matrix $[T_2\circ T_1]_{B',B'}$ by using the product of $[T_2]_{B',B}$ and $[T_1]_{B,B'}$.

   答案： 由于$T_1(1)=0,T_1(x)=1,T_2(x^2)=2x+2,T_2(x^3)=3x^2+6x$，得到

$$[T_1(1)]_B=\begin{bmatrix}0\\0\\0\end{bmatrix},[T_1(x)]_B=\begin{bmatrix}1\\0\\0\end{bmatrix},[T_1(x^2)]_B=\begin{bmatrix}2\\2\\0\end{bmatrix},[T_1(x^3)]_B=\begin{bmatrix}0\\6\\3\end{bmatrix},$$

   因此

$$[T_1]_{B,B'}=\Big[[T_1(1)]_B\quad [T_1(x)]_B\quad [T_1(x^2)]_B\quad [T_1(x^3)]_B\Big]=\begin{bmatrix}0&1&2&0\\0&0&2&6\\0&0&0&3\end{bmatrix}.$$

   由于$T_2(1)=x,T_2(x)=x(2x+1)=2x^2+x,T_2(x^2)=x(2x+1)^2=4x^3+4x^2+x$，得到

$$[T_2(1)]_{B'}=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},[T_2(x)]_{B'}=\begin{bmatrix}0\\1\\2\\0\end{bmatrix},[T_2(x^2)]_{B'}=\begin{bmatrix}0\\1\\4\\4\end{bmatrix},$$

   因此

$$[T_2]_{B',B}=\Big[[T_2(1)]_{B'}\quad [T_2(x)]_{B'}\quad [T_2(x^2)]_{B'}\Big]=\begin{bmatrix}0&0&0\\1&1&1\\0&2&4\\0&0&4\end{bmatrix}.$$

最后，得到

$$[T_2 \circ T_1]_{B',B'} = [T_2]_{B',B}[T_1]_{B,B'} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 4 & 24 \\ 0 & 0 & 0 & 12 \end{bmatrix}.$$

5 10 points

Consider the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose standard matrix is

$$[T] = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

Let $B' = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ be another basis of $\mathbb{R}^2$, where $\boldsymbol{u}_1 = (1, 1), \boldsymbol{u}_2 = (1, 2)$. Find the matrix of T relative to the basis B', i.e., $[T]_{B',B'}$, and show its relation with the standard matrix $[T]$.

答案： 令$B = \{(1, 0), (0, 1)\}$为$\mathbb{R}^2$的标准基底，那么我们已知

$$[T]_{B,B} = [T] = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

另一方面，对于$B' = \{\boldsymbol{u}_1 = (1, 1), \boldsymbol{u}_2 = (1, 2)\}$，我们可以计算出转移矩阵$P_{B \leftarrow B'} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$，以及$P_{B' \leftarrow B} = P_{B \leftarrow B'}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$。因此

$$[T]_{B',B'} = P_{B' \leftarrow B}[T]_{B,B}P_{B \leftarrow B'} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

6 10 points

In $\mathbb{R}^3$, let

$$\boldsymbol{x}_1 = (1, -2, -5), \boldsymbol{x}_2 = (0, 8, 9),$$

and

$$\boldsymbol{y}_1 = (1, 6, 4), \boldsymbol{y}_2 = (2, 4, -1), \boldsymbol{y}_3 = (-1, 2, 5).$$

Determine whether $\text{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2\} = \text{span}\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$, and explain your argument.

答案： 令$W_1 = \text{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$，令$W_2 = \text{span}\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$。如果我们能证明$W_1^{\perp} = W_2^{\perp}$，那么由于$((W_1)^{\perp})^{\perp} = W_1, ((W_2)^{\perp})^{\perp} = W_2$，我们可以得到$W_1 = W_2$。
设$\boldsymbol{x} = (x_1, x_2, x_3) \in W_1^{\perp}$，那么它必须满足

$$\boldsymbol{x} \cdot \boldsymbol{x}_1 = x_1 - 2x_2 - 5x_3 = 0, \quad \boldsymbol{x} \cdot \boldsymbol{x}_2 = 8x_2 + 9x_3 = 0$$

求解此方程组可得通解为$\boldsymbol{x} = r \begin{bmatrix} 11/4 \\ -9/8 \\ 1 \end{bmatrix}, r \in \mathbb{R}$，即$W_1^{\perp} = \text{span}\{\begin{bmatrix} 11/4 \\ -9/8 \\ 1 \end{bmatrix}\}$。
设$\boldsymbol{x} = (x_1, x_2, x_3) \in W_2^{\perp}$，那么它必须满足

$$\boldsymbol{x} \cdot \boldsymbol{y}_1 = x_1 + 6x_2 + 4x_3 = 0, \quad \boldsymbol{x} \cdot \boldsymbol{y}_2 = 2x_1 + 4x_2 - x_3 = 0, \quad \boldsymbol{x} \cdot \boldsymbol{y}_3 = -x_1 + 2x_2 + 5x_3 = 0.$$

求解此方程组可得通解为$\boldsymbol{x} = r \begin{bmatrix} 11/4 \\ -9/8 \\ 1 \end{bmatrix}, r \in \mathbb{R}$，即$W_2^{\perp} = \text{span}\{\begin{bmatrix} 11/4 \\ -9/8 \\ 1 \end{bmatrix}\}$，因此有$W_1^{\perp} = W_2^{\perp}$，即$W_1 = W_2$。

7 15 points

Let V be a finite dimensional vector space. Let $T : V \to V$ be a linear operator satisfying $T^3 = T \circ T \circ T = 4T$. Prove that $\ker(T) + \text{RAN}(T^2) = V$, here $\ker(T)$ denotes the kernel of T, $\text{RAN}(T^2)$ denotes the range of T^2.

答案： 对任何$\boldsymbol{v} \in V$，我们都可以将其表示为

$$\boldsymbol{v} = \frac{1}{4}T^2(\boldsymbol{v}) + (\boldsymbol{v} - \frac{1}{4}T^2(\boldsymbol{v})).$$

显然，$\frac{1}{4}T^2(\boldsymbol{v}) = T^2(\frac{1}{4}\boldsymbol{v}) \in \text{RAN}(T^2)$。另一方面，注意$T(\boldsymbol{v} - \frac{1}{4}T^2(\boldsymbol{v})) = T(\boldsymbol{v}) - \frac{1}{4}T(T^2(\boldsymbol{v})) = T(\boldsymbol{v}) - \frac{1}{4}T^3(\boldsymbol{v})$。由假设$T^3 = 4T$可得$T(\boldsymbol{v} - \frac{1}{4}T^2(\boldsymbol{v})) = T(\boldsymbol{v}) - \frac{1}{4}T^3(\boldsymbol{v}) = T(\boldsymbol{v}) - \frac{1}{4} \times 4T(\boldsymbol{v}) = T(\boldsymbol{v}) - T(\boldsymbol{v}) = \boldsymbol{0}$，从而有$\boldsymbol{v} - \frac{1}{4}T^2(\boldsymbol{v}) \in \ker(T)$。因此，基于以上分析以及$\boldsymbol{v} = \frac{1}{4}T^2(\boldsymbol{v}) + (\boldsymbol{v} - \frac{1}{4}T^2(\boldsymbol{v}))$我们得到$\boldsymbol{v} \in \ker(T) + \text{RAN}(T^2)$对任何$\boldsymbol{v} \in V$成立，从而有$V = \ker(T) + \text{RAN}(T^2)$。

8 15 points

Let $A \in M_{3\times 3}$ be a matrix such that its first row is nonzero, i.e., if A is expressed by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then $(a_{11}, a_{12}, a_{13}) \neq (0,0,0)$. Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \end{bmatrix}$. Assume that $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Compute the nullity of A, and find a basis of Null(A).

答案： 由于矩阵AB的第一列等于A乘以B的第一列，AB的第二列等于A乘以B的第二列，AB的第三列等于A乘以B的第三列，利用假设我们实际上得到

$$A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{0}, A\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \mathbf{0}, A\begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \mathbf{0},$$

因此可得B的三个列向量$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$都属于Null$(A)$。容易验证$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$与$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$线性相关，与$\begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$线性无关，因此rank$(B) = 2$，且Null$(A)$包含两个线性无关的向量$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$。

另一方面，因为$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$，我们可以确定RAN$(B) \subset \ker(A)$。因此得到

$$\text{rank}(B) = \dim(\text{RAN}(B)) \leq \dim(\ker(A)) = \text{nullity}(A);$$

利用以上不等式以及rank–nullity定理，又可以得到

$$\text{rank}(A) + \text{rank}(B) \leq \text{rank}(A) + \text{nullity}(A) = 3.$$

由于rank$(B) = 2$，以上不等式告诉我们rank$(A) \leq 3 - \text{rank}(B) = 1$。又由假设，$A$的第一行不为零向量，因此rank$(A) \geq 1$；因此，最终我们有rank$(A) = 1$，所

以nullity$(A) = 3 - 1 = 2$。此时注意到我们已经证明Null(A)包含两个线性无关的向量$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$，$\begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$，所以$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$，$\begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$是Null$(A)$的一组基底。