Advanced Mathematics Quiz 2

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姓名 ______________ 学号______________

1.(20分)计算极限

$$\lim_{x\to 0}\frac{\mathrm{e}^x\sin x - x(1+x)}{\sin^3 x}$$

(洛也可以)

解：原式 $= \lim\limits_{x\to 0}\dfrac{(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3)(x-\frac{1}{6}x^3)-x-x^2}{x^3}$

$= \lim\limits_{x\to 0}\dfrac{x+x^2+\frac{1}{2}x^3-\frac{1}{6}x^3-x-x^2}{x^3}$

$= \dfrac{1}{3}$

2.(20分)函数$f(x)$满足条件$f(0)=0, f'(0)=2$,求极限

$$\lim_{n\to\infty}\left(f\left(\frac{1}{n^2}\right)+f\left(\frac{2}{n^2}\right)+\cdots+f\left(\frac{n}{n^2}\right)\right)$$

证明 由 $f'(0)=2 \Rightarrow \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\dfrac{f(x)}{x}=2$

$\Rightarrow$ 对 $\forall\varepsilon>0$, $\exists\delta$, 当 $0<|x|<\delta$, 有

$\left|\dfrac{f(x)}{x}-2\right|<\varepsilon$.

$\Rightarrow |f(x)-2x|<\varepsilon\cdot|x|$

令 $\left|f(\frac{1}{n^2})+f(\frac{2}{n^2})+\cdots+f(\frac{n}{n^2})-1-\frac{1}{n}\right|<\varepsilon_0$

$\Rightarrow \left|f(\frac{1}{n^2})-\frac{2}{n^2}+f(\frac{2}{n^2})-\frac{4}{n^2}+\cdots+f(\frac{n}{n^2})-\frac{2n}{n^2}+\frac{n(n+1)}{n^2}-1-\frac{1}{n}\right|<\varepsilon_0$

取 $\varepsilon=\frac{\varepsilon_0}{2}$ 则 $\exists\delta$, 当 $0<|x|<\delta$, 有

$|f(x)-2x|<\frac{\varepsilon_0}{2}|x|$.

取 $\frac{1}{n}<\delta$, 即 $n>\frac{1}{\delta}$, 则

$\left|f(\frac{m}{n^2})-\frac{2m}{n^2}\right|<\frac{\varepsilon_0}{2}\frac{2m}{n^2}$; $m=1,2,\cdots n$.

令 $\left|\frac{\varepsilon_0}{2}(\frac{2}{n^2}+\frac{4}{n^2}+\cdots+\frac{2n}{n^2})\right|<\varepsilon_0$

$\Rightarrow \dfrac{n(n+1)}{2n^2}\varepsilon_0<\varepsilon_0$

$\Rightarrow \dfrac{n^2+n}{2n^2}<1$

即当 $n>1$ 时恒成立.

$\therefore$ 对 $\forall\varepsilon_0>0$, $\exists\delta$, 当 $0<|x|<\delta$, 则

$|f(x)-2x|<\frac{\varepsilon_0}{2}|x|$

取 $N=[\frac{1}{\delta}]+1$, 当 $n>N$ 时, 有

$\left|f(\frac{m}{n^2})-\frac{2m}{n^2}\right|<\frac{\varepsilon_0}{2}\frac{2m}{n^2}$; $m=1,2,\cdots n$

$\Rightarrow \varepsilon_0>\left|\frac{\varepsilon_0}{2}(\frac{2}{n^2}+\frac{4}{n^2}+\cdots+\frac{2n}{n^2})\right|$

$>\left|f(\frac{1}{n^2})-\frac{2}{n^2}+f(\frac{2}{n^2})-\frac{4}{n^2}+\cdots+f(\frac{n}{n^2})-\frac{2n}{n^2}\right|$

$=\left|f(\frac{1}{n^2})+f(\frac{2}{n^2})+\cdots+f(\frac{n}{n^2})-\frac{1}{n}-1\right|$

$\therefore \lim\limits_{n\to\infty}\left(f(\frac{1}{n^2})+f(\frac{2}{n^2})+\cdots+f(\frac{n}{n^2})-\frac{1}{n}\right)=1$

$\Rightarrow \lim\limits_{n\to\infty} f(\frac{1}{n^2})+f(\frac{2}{n^2})+\cdots+f(\frac{n}{n^2})=1$.

证明 $\lim\limits_{n\to\infty}\left(f(\frac{1}{n^2})+f(\frac{2}{n^2})+\cdots+f(\frac{n}{n^2})\right)$

$=\lim\limits_{n\to\infty}\dfrac{f(\frac{1}{n^2})+f(\frac{2}{n^2})+\cdots+f(\frac{n}{n^2})}{\frac{1}{n^2}\cdot n^2}$

$=\lim\limits_{n\to\infty}\frac{1}{n^2}\cdot\left(\dfrac{f(\frac{1}{n^2})}{\frac{1}{n^2}}+\dfrac{f(\frac{2}{n^2})}{\frac{1}{n^2}}+\cdots+\dfrac{f(\frac{n}{n^2})}{\frac{1}{n^2}}\right)$

$=\lim\limits_{n\to\infty}\frac{1}{n^2}\left(\dfrac{f(\frac{1}{n^2})}{\frac{1}{n^2}}+2\cdot\dfrac{f(\frac{2}{n^2})}{\frac{2}{n^2}}+\cdots+n\cdot\dfrac{f(\frac{n}{n^2})}{\frac{n}{n^2}}\right)$

由 $f(0)=0$, $f'(0)=2$

$\Rightarrow f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{f(x)}{x}=2$.

$\Rightarrow \lim\limits_{n\to\infty}\left(f(\frac{1}{n^2})+f(\frac{2}{n^2})+\cdots+f(\frac{n}{n^2})\right)$

此时n没有趋近于无穷，还是有不严谨.

$=\lim\limits_{n\to\infty}\frac{1}{n^2}2\cdot(1+2+\cdots+n)$

$=\lim\limits_{n\to\infty}\frac{1}{n^2}\cdot 2\cdot\frac{(n+1)n}{2}$

$=1$

3.(20分)设$f(x)$在$(-\infty,+\infty)$上有三阶连续导数，$f(0)=1$，$f'(0)=0$.定义

$$g(x)=\begin{cases}\frac{f(x)-1}{x^2}, x\neq 0\\ \\ \frac{f''(0)}{2}, x=0\end{cases}$$

证明：$g(x)$在$(-\infty,+\infty)$有连续导数。

证明：当$x\neq 0$，由$f(x)$在$(-\infty,+\infty)$存在三阶连续导数

且$g(x)=\frac{f(x)-1}{x^2}$ 为 $f(x)$ 的复合函数

$\Rightarrow g(x)$ 在 $(-\infty,0)\cup(0,+\infty)$ 存在连续导数. $g'(x)=\frac{xf'(x)-2(f(x)-1)}{x^3}$

当$x=0$，则由$f(0)=1$，$f'(0)=0$，且$f(x)$存在三阶连续导数

$\Rightarrow f(x)=f(0)+xf'(0)+\frac{1}{2}x^2f''(0)+\frac{1}{6}x^3f'''(\xi)$；

$=1+\frac{1}{2}x^2f''(0)+\frac{1}{6}x^3f'''(\xi)$

其中ξ在0和x之间.

$$\lim_{x\to 0}\frac{g(x)-g(0)}{x}=\lim_{x\to 0}\frac{\frac{f(x)-1}{x^2}-\frac{f''(0)}{2}}{x}$$

$$=\lim_{x\to 0}\frac{\frac{1+\frac{1}{2}x^2f''(0)+\frac{1}{6}x^3f'''(\xi)-1}{x^2}-\frac{f''(0)}{2}}{x}$$

$$=\lim_{x\to 0}\frac{1}{6}f'''(\xi)$$

$$=\frac{1}{6}f'''(0)$$

注意到：

$f'(x)=f'(0)+xf''(0)+\frac{1}{2}x^2f'''(\eta)$

$=xf''(0)+\frac{1}{2}x^2f'''(\eta)$；$\eta$在0与$x$之间.

$$\Rightarrow \lim_{x\to 0}g'(x)=\lim_{x\to 0}\frac{xf'(x)-2(f(x)-1)}{x^3}$$

$$=\lim_{x\to 0}\frac{x\cdot[xf''(0)+\frac{1}{2}x^2f'''(\eta)]-2(\frac{1}{2}x^2f''(0)+\frac{1}{6}x^3f'''(\xi))}{x^3}$$

$$=\lim_{x\to 0}\left(\frac{1}{2}f'''(\eta)-\frac{1}{3}f'''(\xi)\right)$$

$$=\frac{1}{2}f'''(0)-\frac{1}{3}f'''(0)$$

$$=\frac{1}{6}f'''(0)=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}$$

$\Rightarrow g(x)$在$x=0$处导数连续.

$\therefore g(x)$在$(-\infty,+\infty)$导数连续.

4.(20分)设$f(x) = \arcsin x$,

(1)试证明:$(1-x^2)f''(x) = xf'(x)$

(2)求$f^{(2022)}(0)$和$f^{(2023)}(0)$的值。

(3)试给出$f^{(n)}(0)$的值(n为任意自然数)

(1)证明:$f'(x) = \frac{1}{\sqrt{1-x^2}}$; $f''(x) = \frac{x}{(1-x^2)^{3/2}}$

$(1-x^2)f''(x) = (1-x^2)\frac{x}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} = x\cdot f'(x)$.

(2)解 由(1)得:

$(1-x^2)f''(x) - xf'(x) = 0$

两边对x求n次导:$(n\geqslant 2)$

$f^{(n+2)}(x) - x^2 f^{(n+2)}(x) - n2x\cdot f^{(n+1)}(x) - \frac{n(n-1)}{2}\cdot 2f^{(n)}(x) - xf^{(n+1)}(x) - nf^{(n)}(x) = 0$.

取$x=0$,得:

$f^{(n+2)}(0) - \frac{n(n-1)}{2}\cdot 2f^{(n)}(0) - nf^{(n)}(0) = 0$.

$\Rightarrow f^{(n+2)}(0) = n^2 f^{(n)}(0)$.

注意$f'(0)=1$;$f''(0)=0$.

$\Rightarrow f^{(2022)} = 2020^2\cdot 2018^2\cdots\cdot 2^2 f''(0) = 0$;

$f^{(2023)}(0) = 2021^2\cdot 2019^2\cdots\cdot 1^2 f'(0) = [2021!!]^2$.

(3)解 由$f^{(n+2)}(0) = n^2 f^{(n)}(0)$. 且$f'(0)=1$;$f''(0)=0$.

$\Rightarrow$ 当n为奇数,$f^{(n)}(0) = (n-2)^2(n-4)^2\cdots 1^2 f'(0) = [(n-2)!!]^2$

当n为偶数,$f^{(n)}(0) = 0$.

5.(20分)设函数$f(x)$在$[0,1]$上可导,$f(0)=1, f(1)=2$。而且在区间$(0,1)$上$f(x)$无零点或者$f(x)$存在唯一零点。

证明:函数$f^2(x) - 2f'(x)$在$(0,1)$上必有零点存在

证明:设$f(x)$可能的唯一零点为a,即

$f(a)\geqslant 0$ or $f(a)\leqslant 0$;

$f(x)\neq 0$;$x\in(0,a)\cup(a,1)$

1° 当不存在零点,构造函数:

$g(x) = x + \frac{2}{f(x)} - 2$;$x\in(0,1)$

则$g(0) = 0 + \frac{2}{1} - 2 = 0$;$g(1) = 1 + \frac{2}{2} - 2 = 0$.

由于$f(x)\in D[0,1] \Rightarrow f(x)\in C[0,1]$.

$\Rightarrow g(x)\in C[0,1]$

又$g(0) = g(1) = 0 \Rightarrow \exists\xi\in(0,1)$,使$g'(\xi) = 0$.

$\Rightarrow g'(\xi) = 1 - \frac{2}{f^2(\xi)}f'(\xi) = 0$

$\Rightarrow f^2(\xi) - 2f'(\xi) = 0$

2° 当存在零点,由于为唯一零点,记$f(a)=0$, $a\in(0,1)$.

假设$f'(a)\neq 0$.

则$f'(a) = A$.

$\Rightarrow \lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a} = A$.

不妨设$A>0$,则对$\varepsilon = \frac{A}{2}$,$\exists\delta>0$,$0<|x-a|<\delta$时,有:

$\left|\frac{f(x)-f(a)}{x-a} - A\right| < \frac{A}{2}$.

$\Rightarrow \frac{A}{2} < \frac{f(x)-f(a)}{x-a} < \frac{3}{2}A$

令$x<a$,则有$x-a<0$,

$\Rightarrow \frac{3A}{2}(x-a) < f(x)-f(a) < \frac{1}{2}A(x-a)$

$\Rightarrow \frac{3A}{2}(x-a) < f(x) < \frac{1}{2}A(x-a)$ $(f(a)=0)$.

取$x = a-\frac{\delta}{2}$,则

$-\frac{3}{4}A\delta < f(a-\frac{\delta}{2}) < -\frac{1}{4}A\delta$

$\Rightarrow f(a-\frac{\delta}{2}) < 0$.

又$f(0) = 1 > 0$. $f(x)\in C[0, a-\frac{\delta}{2}]$.

$\Rightarrow$ 存在$\xi\in(0, a-\frac{\delta}{2})$,使$f(\xi)=0$. 与题意矛盾.

故假设不成立,$f'(a)=0$.

$\Rightarrow f^2(a) - 2f'(a) = 0$.

综上所述,存在$\xi\in(0,1)$,使$f^2(\xi) - f'(\xi) = 0$.

2° 当存在零点,由于为唯一零点,记$f(a)=0$, $a\in(0,1)$.

假设存在点$x_0\in(0,1)$,使$f(x_0)<0$.

由$f(0)=1$, $f(1)=2$

$\Rightarrow f(0)f(x_0)<0$,$f(1)f(x_0)<0$.

而$f(x)\in C[0,1]$

$\Rightarrow$ 存在$\xi_1\in(0,x_0)$,$\xi_2\in(x_0,1)$,使得

$f(\xi_1) = f(\xi_2) = 0$,$\xi_1\neq\xi_2$

与$f(x)$有唯一零点矛盾.

故$f(x)$不存在$x_0\in(0,1)$,使$f(x_0)<0$.

$\therefore f(x)\geqslant 0$,$x\in[0,1]$

而$f(a)=0 \Rightarrow x=a$为极小值点

$\therefore f'(a)=0$.

$\Rightarrow f^2(a) - 2f'(a) = 0$